Ergodicity for 2D stochastic Burgers equations

Jiang-Lun Wu

Swansea University, UK

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Outline

Based on
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Preliminaries

We are concerned with the ergodicity for 2D stochastic Burgers equation on a bdd domain $\mathcal{O} \subset \mathbb{R}^2$ with smooth $\partial\mathcal{O}$

$$\begin{aligned} d\mathbf{u}(t,\mathbf{x}) &= \nu\Delta\mathbf{u}(t,\mathbf{x})dt - (\mathbf{u}\cdot\nabla\mathbf{u})(t,x)dt + \sum_{k=1}^{\infty} b_k\mathbf{u}(t,\mathbf{x})dB_k(t) \\ \mathbf{u}(t,\mathbf{x}) &= 0, \qquad (t,\mathbf{x}) \in [0,T]\times\partial\mathcal{O} \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}), \qquad \mathbf{x} = (x_1,x_2) \in \mathcal{O}. \end{aligned} \tag{1}$$

Preliminaries

The unknowns : 2D velocity fields $\mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x})) \in \mathbb{R}^2$.

- $T > 0$ is arbitrarily fixed time;
- $\nu > 0$ in (1) stands for the viscosity;
- $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$ be a stochastic basis, the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions;
- $b_k \in \mathbb{R}$ and $(\{B_k(t)\}_{t\in[0,T]})_{k\in\mathbb{N}}$ is a sequence of independent, one-dimensional $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motions.

Preliminaries

Multidimensional (stochastic) Burgers equations appeared in studying turbulence and in modelling the large scale structure of the universe

- V. I. Arnold, S. F. Shandarin, Y. B. Zeldovich: The large scale structure of the universe 1, Geophys. Astrophys. Fluid Dynamics 20 (1982) 111-130.
- S. F. Shandarin, Y. B. Zeldovich: The large scale structure of the universe 2: turbulence, intermittency , structures in a self gravitating medium, Rev. Mod. Phys. 6 (1989) 185-220.
- E. Ben-Naim, S. Y. Chen, G. D. Doolen, S. Redner: Shock like dynamics of inelastic gases, Phy. Rep. Lett. 83 (1999) 4069-4072.
- J. Bec, K. Khanin: Burgers turbulence, Phy. Rep. 447(2007) 1-66.

  [The theory of multidimensional Burgers equation in the non-potential case (i.e., the initial data or the noise do not have gradient form) is a *terra incognita*, there is no global well-posedness in $L^2(\mathcal{O})$ and one even does not know $\mathbb{E}\int_0^1 \|\mathbf{u}(s)\|_1^2 ds < \infty$.]

Known results towards non potential multidimensional Burgers equations

There are just a few papers studying the long-time behaviour for the multidimensional Burgers equations, especially for the stochastic case:

- Kiselev and Ladyzhenskaya [Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957) 655-680] proved the global well-posedness in $L^\infty([0,T]; L^\infty(\mathcal{O})) \cap L^2([0,T]; H_0^1(\mathcal{O}))$.
- Bui [Math. Z. 145 (1975) 69-79] showed that, when the viscosity tends to zero and the initial condition is zero,the convergence of solutions to the inviscid multidimensional Burgers equation.
- Huijiang Zhao and Qingsong Zhao [DCDS-A 41 (2021) 2167-2185] studied the asymptotic behaviour of radially symmetric solutions to 2D Burgers equation.
- Brzezniak, Goldys and Neklyudov [SIAM J. Math. Anal. 46(2014) 871-889] established the global well-posedness for the mild solutions to 3D Burgers equation on $\mathbb{R}^3$ or torus with additive noise, and obtained uniform a priori estimates for the viscosity under Beale-Kato-Majda type condition.

Known results for stochastic potential multidimensional Burgers equations

If the initial data or the force is potential, the analysis for multidimensional Burgers equations becomes comparably simpler. One can apply the Hopf-Cole transformation to reduce the multidimensional Burgers equations either to the heat equations or to the Hamilton-Jacobi equations.

- Iturriaga and Khanin [Commun. Math. Phys. 232 (2003) 377-428] constructed a unique stationary distribution for the inviscid stochastic multidimensional Burgers equations.
- Gomes, Iturriaga, Khanin and Padilla [Moscow Mathematical Journal 5 (2005) 613-631] studied viscosity limit of stationary distributions for the random forced multidimensional Burgers equations.
- Khanin, Ke Zhang [Commun. Math. Phys. 355 (2017) 803-837] studied inviscid stochastic multidimensional Burgers equations and proved that the unique global minimizer is hyperbolic for a class of gradient solutions.

Notations

- $\mathbb{L}^p(\mathcal{O}) = [L^p(\mathcal{O})]^2$, $\mathbb{H}^{m,p} = [H_0^{m,p}(\mathcal{O})]^2$, $\mathbb{H}^m = [H_0^m(\mathcal{O})]^2$.
- the norm in $\mathbb{L}^p(\mathcal{O})$ by $|\cdot|_p$.
- $\langle\cdot,\cdot\rangle$ to be inner product in $\mathbb{L}^2(\mathcal{O}) := \mathbb{H}$,
- Define the linear operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ as $Au := -\Delta u$.
- Denote by $0 < \lambda_1 < \lambda_2 < \cdots <$ the eigenvalues of A, and by $e_1, e_2, \cdots$ the corresponding complete orthonormal system of eigenvectors.
- $D(A^\theta) = \{\mathbf{u} \in \mathbb{H} : \sum_{k=1}^\infty \lambda_k^{2\theta} u_k^2 < \infty\}, \theta \in \mathbb{R}$.
- $A^\theta \mathbf{u} = \Sigma_{k=1}^\infty \lambda_k^\theta u_k e_k$, for $u = \Sigma_{k=1}^\infty u_k e_k$.
- $\|\mathbf{u}\|_{2\theta} := |A^\theta \mathbf{u}|_2 = \left(\Sigma_{k=1}^\infty \lambda_k^{2\theta} |u_k|^2\right)^{1/2}$, $D(A^{\frac{1}{2}}) = \mathbb{H}^1$, $D(A^1) = \mathbb{H}^2$.
- $\mathbb{H}^{-1}$ =the dual space of $\mathbb{H}^1$ and the bilinear operator $B(\mathbf{u},\mathbf{v}) : \mathbb{H}^1 \times \mathbb{H}^1 \to \mathbb{H}^{-1}$ is defined via

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{z}\rangle = \int_{\mathcal{O}} \mathbf{z}(\mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}) \cdot \nabla)\mathbf{v}(\mathbf{x}) d\mathbf{x}$$

for all $\mathbf{z} \in \mathbb{H}^1$.

Definition (Local strong solutions)

Fix stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$. Suppose $\mathbf{u}_0$ is a $D(A^{\frac{1}{2}})$-valued and $\mathcal{F}_0$-measurable random variable. A pair $(\mathbf{u}, \tau)$ is a local strong pathwise solution to (1) if τ is a strictly positive stopping time and $\mathbf{u}(\cdot \wedge \tau)$ is an $\mathcal{F}_t$-adapted process in $D(A^{\frac{1}{2}})$ such that

$$\mathbf{u}(\cdot \wedge \tau) \in C([0,\infty); D(A^{\frac{1}{2}})),\ \mathbf{u} I_{t\leq\tau} \in L^2_{loc}([0,\infty); D(A)),\ \mathbb{P} - a.s.;$$

and, for every $t \geq 0$,

$$\mathbf{u}(t\wedge\tau) + \int_0^{t\wedge\tau} A\mathbf{u}(s) + B(\mathbf{u})(s)ds = \mathbf{u}(0) + \Sigma_{k=1}^{\infty} \int_0^{t\wedge\tau} b_k \mathbf{u}(s) dW_k(s)$$

with equality in $\mathbb{H}$.

Definition (Uniqueness of local strong solutions)

Strong pathwise solutions of (1) are said to be (pathwise) unique up to a random positive time $\tau > 0$, if any two solutions $(\mathbf{u}^1, \tau)$ and $(\mathbf{u}^2, \tau)$ coincide at $t = 0$ on $\tilde{\Omega} := \{\mathbf{u}^1(0) = \mathbf{u}^2(0)\} \subset \Omega$, then

$$P\Big(\mathrm{I}_{\tilde{\Omega}}(\mathbf{u}^1(t \wedge \tau) - \mathbf{u}^2(t \wedge \tau)) = 0; \forall t \geq 0\Big) = 1.$$

Definition (Maximal and global strong solutions)

Let ξ be a positive random variable. We say that the pair $(\mathbf{u}, \xi)$ is a maximal pathwise strong solution, if $(\mathbf{u}, \tau)$ is a local strong pathwise solution for each $\tau < \xi$ and $\sup_{t \in [0,\xi)} \|\mathbf{u}\|_1 = \infty$ almost surely on the set $\{\xi < \infty\}$. Furthermore, if $(\mathbf{u}, \xi)$ is a maximum pathwise strong solution and $\xi = \infty$ a.s., then we say that the solution $\mathbf{u}$ is a global strong solution.

Definition (Local weak solutions)

Suppose that $\mathbf{u}_0$ is a $D(A^{\frac{1}{4}})$ valued, $\mathcal{F}_0$-measurable random variable. A pair $(\mathbf{u}, \tau)$ is a local strong pathwise solution to (1), if τ is a strictly positive stopping time and $\mathbf{u}(\cdot \wedge \tau)$ is an $\mathcal{F}_t$-adapted process in $D(A^{\frac{1}{4}})$ such that

$$\mathbf{u}(\cdot \wedge \tau) \in C([0,\infty); D(A^{\frac{1}{4}})), \mathbf{u} I_{t\leq\tau} \in L^2_{loc}([0,\infty); D(A^{\frac{1}{2}})), \mathbb{P}\text{-a.s. } \omega \in \Omega;$$

and, for every $t \geq 0$,

$$\begin{aligned} & \langle \mathbf{u}(t\wedge\tau), \phi\rangle + \int_0^{t\wedge\tau} \langle \mathbf{u}(s), A\phi\rangle + \int_0^{t\wedge\tau} \langle B(\mathbf{u})(s), \phi\rangle ds \\ = \quad & \langle \mathbf{u}(0), \phi\rangle + \Sigma_{k=1}^{\infty} \int_0^{t\wedge\tau} b_k \langle \mathbf{u}(s,x), \phi\rangle dW_k(s), \end{aligned}$$

for all $t \in \mathbb{R}^+$ and $\phi \in D(A)$.

Definition (Uniqueness of local weak solutions)

1. Weak pathwise solutions of (1) are said to be (pathwise) unique up to a random positive time $\tau > 0$ if any two solutions $(\mathbf{u}^1, \tau)$ and $(\mathbf{u}^2, \tau)$ coincide at $t = 0$ on the event $\tilde{\Omega} := \{\mathbf{u}^1(0) = \mathbf{u}^2(0)\} \subset \Omega$, then

$$P\Big(\mathrm{I}_{\tilde{\Omega}}(\mathbf{u}^1(t \wedge \tau) - \mathbf{u}^2(t \wedge \tau)) = 0; \forall t \geq 0\Big) = 1.$$

Definition (Maximal and global weak solutions)

(i) Let ξ be a positive random variable. We say the pair $(\mathbf{u}, \xi)$ is a maximal pathwise weak solution, if $(\mathbf{u}, \tau)$ is a local weak pathwise solution for each $\tau < \xi$ and

$$\sup_{t\in[0,\xi)} \|\mathbf{u}\|_{\frac{1}{2}} = \infty$$

almost surely on the set $\{\xi < \infty\}$.

(ii) If $(\mathbf{u}, \xi)$ is a maximum pathwise strong solution and $\xi = \infty$ a.s., then we say that the solution $\mathbf{u}$ is a global weak solution.

Main Theorem

Theorem-Existence and uniqueness of invariant measures

Given $\mathcal{F}_0$-measurable initial data $\mathbf{u}_0 \in D(A^{\frac{1}{4}})$ with $\mathbb{E}\|\mathbf{u}_0\|_{\frac{1}{2}}^2 < \infty$, then under the condition $b := \sum_k b_k^2 < 2\lambda_1 + 1$, there exists a unique invariant measure to the 2D stochastic Burgers equation (1).

A useful transformation

Write $W(t) := \sum_{k=1}^{m} b_k B_k(t)$ for short with $t \in [0, \infty)$. Let $\delta = b + \epsilon - 1$. We define

$$\alpha(t) := \exp\Big(- W(t) + \frac{t(b+\epsilon)}{2} \Big).$$

Next, set $\mathbf{v} := \alpha \mathbf{u}$, then equation (1) is equivalent to

$$\begin{aligned} & d\mathbf{v}(t,\mathbf{x}) = \Delta \mathbf{v}(t,\mathbf{x})dt - \alpha^{-1}[(\mathbf{v}\cdot\nabla\mathbf{v})(t,\mathbf{x})]dt + 2^{-1}\delta \mathbf{v}dt, \text{on } [0,T]\times\mathcal{O}, \\ & \mathbf{v}(t,\mathbf{x}) = 0, \text{on } [0,T]\times\partial\mathcal{O}, \qquad (2) \\ & \mathbf{v}(0,\mathbf{x}) = \mathbf{u}(0,\mathbf{x}), \quad \mathbf{x} = (x_1,x_2) \in \mathcal{O}. \end{aligned}$$

Proposition-Local well-posedness of strong solutions

Suppose $\mathbf{u}_0$ is an $D(A^{\frac{1}{2}})$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|\mathbf{u}_0\|_1^2 < \infty$. Then there exists a unique local strong solution $\mathbf{v}$ to equations (2) on the time interval $[0, T]$.

Proof.

Let $\mathbf{v}_n$ be the local solution to the Galerkin approximation of (2). By standard energy estimate, we can obtain

$$\begin{aligned} & \|\mathbf{v}_n(t\wedge\tau_{n,\omega})\|_1^2 + \int_0^t \|\mathbf{v}_n(s\wedge\tau_{n,\omega}))\|_2^2 ds \\ \leq \quad & \|\mathbf{u}(0)\|_1^2 + c(\varepsilon,\delta,T)\int_0^t (1+\|\mathbf{v}_n(s)\|_1^2)^{\frac{5}{2}} ds. \end{aligned}$$

For $t \leq t^* = \frac{2}{3c(\delta,T)(1+\|u_0\|_1^2)^{\frac{3}{2}}}$, by comparison theorem, one can show

$$\|\mathbf{v}_n(t\wedge\tau_{n,\omega}))\|_1^2 \quad \leq \quad \frac{1+\|\mathbf{u}_0\|_1^2}{\left(1-\frac{3}{2}tc(\varepsilon,\delta,T)(1+\|\mathbf{u}_0\|_1^2)^{\frac{3}{2}}\right)^{\frac{2}{3}}} - 1,$$

which implies the local existence of $\mathbf{v}$. The proof of uniqueness of $\mathbf{v}$ is routine. □

Utilising the maximum principle (see e.g. Evans: Partial Differential Equation, 2nd edition, 2016), one can show the following

Lemma-Maximum estimates for $\mathbf{v}_n$

If $\mathbf{v}_n$ is a solution of the Galerkin approximations of Burgers equation (2) on the time interval $[0, t]$ then

$$\sup_{s\in[0,t]} |\mathbf{v}_n(s)|_\infty \quad \leq \quad e^{\frac{\delta}{2}t}|\mathbf{v}_n(0)|_\infty. \tag{3}$$

Based on the local existence theorem, one can establish the global well-posedness of the strong solutions to the (1) with the help of (3).

Theorem-Global well-posedness of strong solutions

For any $\mathcal{F}_0$-adapted initial value $\mathbf{u}_0 \in D(A^{\frac{1}{2}})$ with $\mathbb{E}\|\mathbf{u}_0\|_1^2 < \infty$ and any $T > 0$, there exists a unique global strong solution $\mathbf{v}$ to (2) satisfying

$$\begin{aligned} & \sup_{t\in[0,T]} \|\mathbf{v}(t)\|_1^2 + \int_0^T \|\mathbf{v}(s)\|_2^2 ds \\ \leq \quad & \|\mathbf{u}(0)\|_1^2 \exp(c\|\mathbf{u}(0)\|_1^2 \int_0^T \exp(W(s) - s)ds) \exp(\delta t) < \infty, \end{aligned}$$

$\mathbb{P} - a.s.\ \omega \in \Omega$.

which then implies

the existence and uniqueness of strong solutions $\mathbf{u}$ to 2D stochastic Burgers equation (1).

Proof.

Taking inner product of local solution $\mathbf{v}_n$ in $D(A^{\frac{1}{2}})$ and by maximum estimate (3),

$$\begin{aligned} & d\|\mathbf{v}_n\|_1^2 + 2\|\mathbf{v}_n\|_2^2 dt - \delta\|\mathbf{v}_n\|_1^2 dt \\ = \; & 2\alpha^{-1}\int_D (\mathbf{v}_n\cdot\nabla)\mathbf{v}_n \Delta \mathbf{v}_n dxdt \\ \leq \; & 2\alpha^{-1}|\mathbf{v}_n|_\infty \|\mathbf{v}_n\|_1\|\mathbf{v}_n\|_2 dt \\ \leq \; & \varepsilon\|\mathbf{v}\|_2^2 dt + c(\varepsilon)\alpha^{-2}(t)\exp(\delta t)\|\mathbf{u}_0\|_1^2\|\mathbf{v}_n\|_1^2 dt. \end{aligned}$$

By the Gronwall inequality , we obtain

$$\begin{aligned} & \|\mathbf{v}_n(t)\|_1^2 + \int_0^t \|\mathbf{v}_n(s)\|_2^2 ds \\ \leq \; & \|\mathbf{u}_0\|_1^2 \exp\left(c\|\mathbf{u}_0\|_1^2 \int_0^t \exp(W(s)-s)ds \right)\exp(\delta t). \end{aligned}$$

Similar to the case of strong solutions, one can establish the following

Theorem-Global well-posedness of weak solutions

For any $\mathcal{F}_0$ adapted initial value $\mathbf{u}_0 \in D(A^{\frac{1}{4}})$ with $E\|\mathbf{u}_0\|^2_{\frac{1}{2}} < \infty$ and any $T > 0$, there exists a unique global weak solution $\mathbf{v}$ to (2).

which implies

the existence and uniqueness of weak solution $\mathbf{u}$ to 2D stochastic Burgers equation (1).

Theorem-Maximum estimates for the strong solutions

For any $\mathcal{F}_0$ adapted initial value $\mathbf{u}_0 \in D(A^{\frac{1}{2}})$ with $\mathbb{E}\|\mathbf{u}_0\|_1^2 < \infty$. Then for any $t \in [0, \infty)$, the unique global strong solution $\mathbf{v}$ to (2) satisfies

$$|\mathbf{v}(t)|_\infty \leq \exp\left(\frac{\delta t}{2}\right)|\mathbf{v}(0)|_\infty = \exp\left(\frac{\delta t}{2}\right)|\mathbf{u}_0|_\infty, \quad \mathbb{P} - a.s.\ \omega \in \Omega,$$

which implies

$$|\mathbf{u}(t)|_\infty \leq \exp\left(W(t) - \frac{t}{2}\right)|\mathbf{u}_0|_\infty \to 0, \quad as\ t \to \infty,\ \mathbb{P} - a.s.\ \omega \in \Omega.$$

Proof.

Similar to the proof of global well-posedness of strong solution $\mathbf{u}$, by virtue of Aubin-Lions Lemma, we can extend the maximum estimates of $\mathbf{v}_n$ to $\mathbf{v}$. Then note $\mathbf{u}(t) = \mathbf{v}(t)\alpha^{-1}(t) = \mathbf{v}(t)\exp(W(t) - \frac{(b+\epsilon)}{2})$, the result follows. □

Theorem-Exponential decay

For any $\mathcal{F}_0$ adapted initial value $\mathbf{u}_0 \in D(A^{\frac{1}{4}})$ with $\mathbb{E}\|\mathbf{u}_0\|^2_{\frac{1}{2}} < \infty$, under the condition $b < 2\lambda_1 + 1$, the unique weak solution $\mathbf{u}(t)$ and $\mathbf{v}(t)$ to (1) and (2) respectively, on $t \in [0, \infty)$ satisfy

$$\|\mathbf{u}(t)\|^2_{\frac{1}{2}} \leq c(\omega, \|\mathbf{u}_0\|_{\frac{1}{2}}) \exp(-\lambda t), \mathbb{P} - a.s. \omega \in \Omega,$$

where λ is a positive constant belonging to $(0, 2\lambda_1 + 1)$, as well as

$$\|\mathbf{v}(t)\|^2_{\frac{1}{2}} \leq c(\omega, \|\mathbf{u}_0\|_{\frac{1}{2}}) \exp(-\lambda_0 t), \mathbb{P} - a.s. \omega \in \Omega,$$

where λ_0 is some positive constant.

Proof.

Taking inner product of (2) in $D(A^{\frac{1}{4}})$ and integrating over $(0, t)$ yields

$$\begin{aligned}\frac{1}{2}\|\mathbf{v}(t)\|_{\frac{1}{2}}^2 + \int_0^t \|\mathbf{v}(s)\|_{\frac{3}{2}}^2 ds &\leq \frac{1}{2}\|\mathbf{u}_0\|_{\frac{1}{2}}^2 + \frac{\delta}{2}\int_0^t \|\mathbf{v}(s)\|_{\frac{1}{2}}^2 ds \\ &\quad + \int_0^t \alpha^{-1}(s)|\mathbf{v}(s)|_\infty \|\mathbf{v}(s)\|_1^2 ds.\end{aligned}$$

On the other hand, one can derive for $t \geq 1$

$$\begin{aligned}\int_0^t \alpha^{-1}(s)|\mathbf{v}(s)|_\infty \|\mathbf{v}(s)\|_1^2 ds &\leq \int_1^t \alpha^{-1}(s)|\mathbf{v}(1)|_\infty \|\mathbf{v}(s)\|_1^2 ds \\ &\quad + \int_0^1 \alpha^{-1}(s)\|\mathbf{v}(s)\|_{\frac{1}{2}}^{\frac{1}{2}} \|\mathbf{v}(s)\|_{\frac{3}{2}}^{\frac{1}{2}} \|\mathbf{v}(s)\|_1^2 ds (< \infty),\end{aligned}$$

taking $t = 1$ as the new initial time for the system (2). Applying the maximum estimate (4) for **v**, the results of this theorem follows. □

Theorem-Moment estimates

Given any deterministic initial value $\mathbf{u}_0 \in D(A^{\frac{1}{2}})$. If $2b < 1$, then

$$\sup_{T\in[0,\infty)} \frac{\int_0^T \mathbb{E}\log(\|\mathbf{u}(t)\|_1^2 + 1)dt}{T} < \infty.$$

Proof.

Taking inner product of $\mathbf{v}_n$ in $\mathbb{H}^1$ yields

$$\frac{1}{2}d\|\mathbf{v}\|_1^2 + (\lambda_1 - \varepsilon - \delta/2)\|\mathbf{v}\|_1^2 dt \le c\alpha^{-2}|\mathbf{v}|_\infty^2\|\mathbf{v}\|_1^2 dt$$

$$|\mathbf{v}(t)|_\infty \le \exp\left(\frac{\delta t}{2}\right)|\mathbf{u}_0|_\infty, \mathbb{P}-a.s. \text{ and } \int_0^\infty \exp(\delta s)\,\mathbb{E}\alpha^{-2}(s)ds < \infty.$$

The result follows by logarithmic energy estimate $d\log(\|\mathbf{u}(t)\|_1^2 + 1)$. □

Let $C_b(D(A^{\frac{1}{4}}))$(resp. $M_b(D(A^{\frac{1}{4}}))$) be the set of all real valued bounded continuous (resp. bounded, Borel measurable) functions on $D(A^{\frac{1}{4}})$. One can show that $\mathbf{u}(t,0,\mathbf{u}_0) =: \mathbf{u}(t,\mathbf{u}_0)$ is Markov in the following sense: For every $F \in M_b(D(A^{\frac{1}{4}}))$ and all $s,t \in [0,T]$, $0 \le s \le t \le T$

$$\mathbb{E}[F(\mathbf{u}(t,\mathbf{u}_0))|\mathcal{F}_s](\omega) = \mathbb{E}[F(\mathbf{u}(t,s,\mathbf{u}(s)))] \ \text{ for } \mathbb{P} - a.s.\ \omega \in \Omega.$$

where $\mathcal{B}(D(A^{\frac{1}{4}}))$ denotes the totality of Borel subsets of $D(A^{\frac{1}{4}})$. For $t \ge 0$, define the Markov semigroup

$$P_t\phi(\mathbf{u}_0) := \mathbb{E}\phi(\mathbf{u}(t,\mathbf{u}_0)) = \int_{D(A^{\frac{1}{4}})} \phi(\xi) P_t(\mathbf{u}_0, d\xi) \tag{4}$$

which maps $M_b(D(A^{\frac{1}{4}}))$ into itself. If there exists a probability measure μ on $D(A^{\frac{1}{4}})$such that for any $\phi \in C_b(D(A^{\frac{1}{4}}))$ we have

$$\int_{D(A^{\frac{1}{4}})} P_t\phi(x)\mu(dx) = \int_{D(A^{\frac{1}{4}})} \phi(x)\mu(dx)$$

we say μ is an invariant measure for P_t.

For $\mathbf{u}_0 \in D(A^{\frac{1}{4}})$, let $\mathbf{u}$ be the unique weak solution to (1). Define

$$\tau_k(\mathbf{u}_0) := \inf_{t \geq 0}\{t : \int_0^t \|\mathbf{u}(s, \mathbf{u}_0)\|_1^2 ds \geq k\}$$

$$\sigma_j(\mathbf{u}_0) := \inf_{t \geq 0}\{t : t\|\mathbf{u}(t, \mathbf{u}_0)\|_1^2 ds \geq j\}.$$

Furthermore, define

$$\tau_k(\mathbf{u}_0, \tilde{\mathbf{u}}_0) := \tau_k(\mathbf{u}_0) \wedge \tau_k(\tilde{\mathbf{u}}_0)$$

$$\sigma_j(\mathbf{u}_0, \tilde{\mathbf{u}}_0) := \sigma_j(\mathbf{u}_0) \wedge \sigma_j(\tilde{\mathbf{u}}_0).$$

Proposition-Local Lipschitz continuity

Let $k \geq 2e$, and $t > 0$. Assume $\mathbf{u}_1$ and $\mathbf{u}_2$ are the solutions of (1) with initial data $\mathbf{u}_0, \tilde{\mathbf{u}}_0 \in D(A^{\frac{1}{4}})$ respectively. Then we have

$$\mathbb{E} \sup_{s \in [0, t \wedge \tau_k(\mathbf{u}_0, \tilde{u}_0)]} \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{\frac{1}{2}}^2 \leq c\mathbb{E}\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{\frac{1}{2}}^2 \exp\Big(1 + \mathbf{b}\Big)^2 tk.$$

where c is a positive constant independent of t and k .

Proof.

Set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. Then we apply the Itô formula to $\|\mathbf{u}\|_{\frac{1}{2}}^2$ and obtain that

$$\begin{aligned} d\|\mathbf{u}\|_{\frac{1}{2}}^2 + 2\|\mathbf{u}\|_{\frac{3}{2}}^2 dt &= -2\langle \mathbf{u} \cdot \nabla \mathbf{u}_1, A^{\frac{1}{2}}\mathbf{u}\rangle dt - 2\langle \mathbf{u}_2 \cdot \nabla \mathbf{u}, A^{\frac{1}{2}}\mathbf{u}\rangle dt \\ &\quad + 2\sum_{k=1}^{\infty} \|\mathbf{u}\|_{\frac{1}{2}}^2 b_k dB_k(t) + \sum_{k=1}^{\infty} b_k^2 \|\mathbf{u}\|_{\frac{1}{2}}^2 dt. \end{aligned}$$

Define

$$\mathbf{s}_m := \inf_{t\geq 0}\{t : \|\mathbf{u}(t)\|^2_{\frac{1}{2}} \geq m\} \quad .$$

By standard argument,

$$\begin{aligned} & \mathbb{E} \sup_{s\in[0,t\wedge\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\wedge\mathbf{s}_m]} \|\mathbf{u}(s)\|^2_{\frac{1}{2}} + \int_0^{t\wedge\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\wedge\mathbf{s}_m} \|\mathbf{u}(s)\|^2_{\frac{3}{2}} ds \\ \leq \; & 2\mathbb{E} \sup_{s\in[0,t\wedge\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\wedge\mathbf{s}_m)]} \int_0^s \|\mathbf{u}(t)\|^2_{\frac{1}{2}} \left(\|\mathbf{u}_1\|^2_1 + \|\mathbf{u}_2\|^2_1\right) ds \\ & + cb^2\mathbb{E}\left(\int_0^{t\wedge\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\wedge\mathbf{s}_m} \|\mathbf{u}(s)\|^2_{\frac{1}{2}} ds\right) \\ & + b\mathbb{E}\int_0^{t\wedge\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\wedge\mathbf{s}_m} \|\mathbf{u}(s)\|^2_{\frac{1}{2}} ds + \mathbb{E}\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|^2_{\frac{1}{2}}. \end{aligned}$$

The result follows by stochastic Gronwall inequality and Fatou's Lemma. □

Theorem-Feller property

The Markov semigroup P_t associated to the 2D stochastic Burgers equation (1) with initial data $\mathbf{u}_0 \in D(A^{\frac{1}{4}})$ is Feller on $D(A^{\frac{1}{4}})$, that is P_t maps $C_b(D(A^{\frac{1}{4}}))$ into itself.

Proof.

Fix $t > 0$ and $\phi \in C_b\left(D(A^{\frac{1}{4}})\right)$ and $\mathbf{u}_0 \in D\left(A^{\frac{1}{4}}\right)$. Let $B_{D(A^{\frac{1}{4}})}(\delta, \mathbf{u}_0)$ be the ball of center $\mathbf{u}_0$ and radius δ in $D\left(A^{\frac{1}{4}}\right)$. Given $\varepsilon > 0$, we need to find $\delta \in (0, 1)$ such that

$$|P_t\phi(\mathbf{u}_0) - P_t\phi(\tilde{\mathbf{u}}_0)| = |\mathbb{E}\left(\phi(\mathbf{u}(t, \mathbf{u}_0)) - \phi(\mathbf{u}(t, \tilde{\mathbf{u}}_0))\right)| < \varepsilon, \tag{5}$$

holds for any $\tilde{\mathbf{u}}_0 \in B_{D(A^{\frac{1}{4}})}(\delta, \mathbf{u}_0)$.

Observe that for any $\tilde{\mathbf{u}}_0 \in B_{D(A^{\frac{1}{4}})}(1, \mathbf{u}_0)$ and $k, j \geq 1$, we have

$$
\begin{aligned}
& |\mathbb{E}\left(\phi(\mathbf{u}(t,\mathbf{u}_0)) - \phi(\mathbf{u}(t,\tilde{\mathbf{u}}_0))\right)| \\
\leq \; & |\mathbb{E}\left(\phi(\mathbf{u}(t,\mathbf{u}_0)) - \phi(\mathbf{u}(t,\tilde{\mathbf{u}}_0))\right) I_{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0)>t}| \\
& + |\mathbb{E}\left(\phi(\mathbf{u}(t,\mathbf{u}_0)) - \phi(\mathbf{u}(t,\tilde{\mathbf{u}}_0))\right) I_{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\leq t}| \\
\leq \; & |\mathbb{E}\left(\phi(u(t,\mathbf{u}_0)) - \phi(u(t,\tilde{\mathbf{u}}_0))\right) I_{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0)>t}| + 2|\phi|_\infty \mathbb{P}\{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0) \leq t\} \\
\leq \; & |\mathbb{E}\left(\phi(\mathbf{u}(t,\mathbf{u}_0)) - \phi(\mathbf{u}(t,\tilde{\mathbf{u}}_0))\right) I_{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0)>t} I_{\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\geq t}| + 2|\phi|_\infty \mathbb{P}\{\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0) \\
& + 2|\phi|_\infty \mathbb{P}\{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0) \leq t\} \\
= \; & : I_1 + I_2 + I_3.
\end{aligned}
$$

By (**??**), $\lim_{k\to\infty} I_2 = 0$ and $\lim_{j\to\infty} I_3 = 0$.

Set

$$Lip\left(D(A^{\frac{1}{4}})\right) := \left\{\tilde{\phi} \in C_b\left(D(A^{\frac{1}{4}})\right) : \tilde{\phi} \text{ is Lipschitz continuous on } D\left(A^{\frac{1}{4}}\right)\right\}.$$

Noticing on the set $\{\sigma_j(\mathbf{u}_0, \tilde{\mathbf{u}}_0) > t\}$, one has $\mathbf{u}(t, \mathbf{u}_0)$, $\mathbf{u}(t, \tilde{\mathbf{u}}_0) \in B_{D(A^{\frac{1}{2}})}\left(\frac{j}{t}, \mathbf{0}\right)$. Hence, for any $j, k > 1$, one can choose $\tilde{\phi} \in Lip\left(D(A^{\frac{1}{4}})\right)$ such that

$$\begin{aligned}
I_1 &\leq 2 \sup_{u \in B_{D(A^{\frac{1}{2}})}(\frac{j}{t}, \mathbf{0})} |\phi(\mathbf{u}) - \tilde{\phi}(\mathbf{u})| + \left|\mathbb{E}\left(\tilde{\phi}\left(\mathbf{u}(t, \mathbf{u}_0)\right) - \tilde{\phi}\left(\mathbf{u}(t, \tilde{\mathbf{u}}_0)\right)\right) I_{\tau_k(\mathbf{u}_0, \tilde{\mathbf{u}}_0) \geq t}\right| \\
&\leq \frac{\varepsilon}{4} + L\mathbb{E}\left|\mathbf{u}\left(t \wedge \tau_k(\mathbf{u}_0, \tilde{\mathbf{u}}_0), \mathbf{u}_0\right) - \mathbf{u}\left(t \wedge \tau_k(\mathbf{u}_0, \tilde{\mathbf{u}}_0), \tilde{\mathbf{u}}_0\right)\right| \\
&\leq \frac{\varepsilon}{4} + L\exp\left(C(C_0, t, k)\right) \|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{\frac{1}{2}} < \frac{\varepsilon}{2},
\end{aligned}$$

where the last estimate follows by $\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{\frac{1}{2}}$ small enough. Then the Feller property (5) follows by the estimates from I_1 to I_3. □

Theorems on Feller property and the decay estimates together imply that the Dirac measure at zero δ_0 is indeed an invariant measure to (1). Hence, we get the existence result

Theorem-Existence of invariant measures

Under condition $b < 2\lambda_1 + 1$, there exists an invariant measure to (1) by the fact that δ_0 is an invariant measure to (1) (which is an ergodic measure).

In fact the Theorems on Feller property and moment estimate via Krylov-Bogoliubov procedure together imply the existence of invariant measures under the condition $2b < 1$ which is stronger than the Theorem. But this method can be extended to additive case.

By the global well-posedness for weak and strong solutions to (1), for a weak solution $\mathbf{u} \in C([0,T]; D(A^{\frac{1}{4}})) \cap L^2([0,T]; D(A^{\frac{3}{4}}))$, we get $\mathbf{u} \in C([t_0,T]; D(A^{\frac{1}{2}}))$ for $t_0 \in (0,T)$. Combining with decay estimates in $D(A^{\frac{1}{4}})$, we have exponential decay estimate for $\mathbf{u} \in D(A^{\frac{1}{2}})$

Theorem-Exponential decay in $D(A^{\frac{1}{2}})$

For any $\mathcal{F}_0$-adapted initial value $\mathbf{u}_0 \in D(A^{\frac{1}{4}})$ with $\mathbb{E}\|\mathbf{u}_0\|_{\frac{1}{2}}^2 < \infty$. Then under the condition $b < 2\lambda_1 + 1$, the unique weak solution $\mathbf{u}(t)$ and $\mathbf{v}(t)$ to (1) and (2), respectively, on $t \in [0,\infty)$ satisfy

$$\|\mathbf{u}(t)\|_1^2 \leq c(\omega, \|\mathbf{u}_0\|_{\frac{1}{2}}) \exp(-\lambda t), \ \mathbb{P} - a.s.\, \omega \in \Omega$$

where λ is a positive constant such that $\lambda \in (0, 2\lambda_1 + 1)$, and

$$\|\mathbf{v}(t)\|_1^2 \leq c(\omega, \|\mathbf{u}_0\|_{\frac{1}{2}}) \exp(-\lambda_0 t), \ \mathbb{P} - a.s.\, \omega \in \Omega$$

where λ_0 is a positive constant.

In this part, we will use asymptotic coupling method in
N. Glatt-Holtz, J.C. Mattingly, G. Ricards: *On uniqueness ergodicity in nonlinear stochastic partial differential equations,* J. Stat. Phys. 166 (2017) 618–649.
to prove the uniqueness of invariant measures. For distances ρ and $\tilde{\rho}$ on Hilbert space H, $\tilde{\rho} \leq c\rho$ for some constant c, we define

$$D_{\tilde{\rho}} := \left\{(u,v) \in H^{\mathbb{N}} \times H^{\mathbb{N}} : \lim_{n\to\infty} \tilde{\rho}(u_n, \mathbf{v}_n) = 0\right\},$$

$$\mathcal{G}_{\tilde{\rho}} := \left\{\phi \in C_b(H) : \sup_{x\neq y} \frac{|\phi(x)-\phi(y)|}{\tilde{\rho}(x,y)} < \infty\right\}.$$

Theorem-Asymptotic coupling [Glatt-Holtz, Mattingly, Ricards]

Suppose that $\mathcal{G}_{\tilde{\rho}}$ determines measures on (H,ρ) and assume that $D_{\tilde{\rho}}$ is a measurable subset of $H^{\mathbb{N}} \times H^{\mathbb{N}}$. If $H_0 \subset H$ is a measurable set such that for each pair $\mathbf{u}_0, \mathbf{v}_0 \in H_0$ there exists an element $\mu_{\mathbb{N}\times\mathbb{N}} \in \Pi(\delta_{\mathbf{u}_0} P^{\mathbb{N}}, \delta_{\mathbf{v}_0} P^{\mathbb{N}})$ with $\mu_{\mathbb{N}\times\mathbb{N}}(D_{\tilde{\rho}}) > 0$, then there exists at most one ergodic invariant measure μ with $\mu(H_0) > 0$.

Theorem-Uniqueness of invariant measures for 2D stochastic Burgers equation

If $b < 2\lambda_1 + 1$, then there is a unique invariant measure to (1). Consequently, the (ergodic) Dirac measure δ_0 as the unique invariant measure ensures the ergodicity for (1).

Proof. We apply the above asymptotic coupling theorem to the case of (1) with $(H, \rho) = \left(D(A^{\frac{1}{4}}\right), \|\cdot\|_{\frac{1}{2}})$ and $\tilde{\rho} = |\cdot|_2$. By the exponential decay estimate established in $D(A^{\frac{1}{2}})$ for weak solution, there exists a big enough constant $R > 0$ such that

$$\mathbb{P}(B_R) := \mathbb{P}\left(\sup_{t\in[0,\infty)} |\mathbf{u}(t)|_2^2 + \int_0^\infty \|\mathbf{u}(s)\|_1^2 ds \le R\right) > 0. \tag{6}$$

For any $\mathbf{u}_0, \tilde{\mathbf{u}}_0 \in D(A^{\frac{1}{4}})$, let $\mathbf{u}$ to be the unique weak pathwise solution to (1) with $\mathbf{u}(0) = \mathbf{u}_0$ and $\tilde{\mathbf{u}}$ solve the following system,

$$\begin{aligned} d\tilde{\mathbf{u}}(t,\mathbf{x}) &= \Delta\tilde{\mathbf{u}}(t,\mathbf{x})dt - I_{\{\eta_k > t\}}|\mathbf{u} - \tilde{\mathbf{u}}|_2\tilde{\mathbf{u}}(t,\mathbf{x})dt - (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}})(t,\mathbf{x})dt \\ &\quad + \sum_{k=1}^{\infty} b_k\tilde{\mathbf{u}}(t,\mathbf{x})dB_k(t), \;\; \text{on } (0,T] \times \mathcal{O}, \\ \tilde{\mathbf{u}}(t,\mathbf{x}) &= 0, \qquad\quad \text{on } [0,T] \times \partial\mathcal{O}, \\ \tilde{\mathbf{u}}(0,\mathbf{x}) &= \tilde{\mathbf{u}}_0(\mathbf{x}), \quad \mathbf{x} = (x_1,x_2) \in \mathcal{O}. \end{aligned}$$

where $\eta_k := \inf_{t\in[0,\infty)} \{t : |\mathbf{u}(t) - \tilde{\mathbf{u}}(t)|_2 \geq k\}$ and $k > 0$ are fixed positive parameters which we will be determined below as a function of $\mathbf{u}_0$ and $\tilde{\mathbf{u}}_0$.

Let $\tilde{\mathbf{v}} = \alpha\tilde{\mathbf{u}}$. Then we have the equivalent form of the equations for $\tilde{\mathbf{u}}$ as follows

$$\begin{aligned} d\tilde{\mathbf{v}}(t,\mathbf{x}) &= \Delta\tilde{\mathbf{v}}(t,\mathbf{x})dt - I_{\{\tau_k > t\}}\alpha^{-1}|\mathbf{v}-\tilde{\mathbf{v}}|_2\tilde{\mathbf{v}}(t,\mathbf{x})dt \\ &\quad -\alpha^{-1}(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}})(t,\mathbf{x})dt + 2^{-1}\delta\tilde{\mathbf{v}}(t,\mathbf{x})dt, \quad \text{on } (0,T]\times\mathcal{O}, \\ \tilde{\mathbf{v}}(t,\mathbf{x}) &= 0, \qquad \text{on } [0,T]\times\partial\mathcal{O}, \\ \tilde{\mathbf{v}}(0,\mathbf{x}) &= \tilde{\mathbf{u}}_0(\mathbf{x}), \quad \mathbf{x} = (x_1,x_2)\in\mathcal{O}, \end{aligned}$$

Fix $T > 0$ and take $t_n = nT$. Define the measures $\mu_{\mathbb{N}}$ and $\nu_{\mathbb{N}}$ on $D^{\mathbb{N}}\left(A^{\frac{1}{4}}\right)$ to be, respectively, the laws of the random vectors

$$(\mathbf{u}(t_1,\mathbf{u}_0),\mathbf{u}(t_2,\mathbf{u}_0),\ldots) \quad \text{and} \quad (\tilde{\mathbf{u}}(t_1,\tilde{\mathbf{u}}_0),\tilde{\mathbf{u}}(t_2,\tilde{\mathbf{u}}_0),\ldots).$$

By Girsanov's Theorem, one can prove $\mu_{\mathbb{N}}$ is mutually absolutely continuous with respect to $\nu_{\mathbb{N}}$.

Let $\psi = \mathbf{v} - \tilde{\mathbf{v}}$, then η satisfies

$$d\psi = \Delta\psi dt - \alpha^{-1}\psi \cdot \nabla \mathbf{v} dt - \alpha^{-1}\tilde{\mathbf{v}} \cdot \nabla\psi dt + 2^{-1}\delta\psi dt + \alpha^{-1} I_{\{\tau_k > t\}} |\psi|_2 \tilde{\mathbf{v}} dt.$$

On $[0, \eta_k]$, we have

$$\begin{aligned} |\psi(t)|_2^2 &\leq |\psi(0)|_2^2 \exp\left(-2(1-\varepsilon)\lambda_1 t + \delta t\right) \\ &\quad \times \exp c(\varepsilon)\left(\int_0^\infty \alpha^{-2}(\|\mathbf{v}\|_1^2 + \|\tilde{\mathbf{v}}\|_1^2) ds\right) < \infty. \qquad (7) \end{aligned}$$

where the last inequality follows by exponential decay estimate for $\mathbf{v}$ and $\tilde{\mathbf{v}}$ (in fact $\tilde{\mathbf{v}}$ has better regularity than $\mathbf{v}$). (7) implies $B_R \subset \{\eta_k = \infty\}$ and $\mathbb{P}\{\eta_k = \infty\} > 0$.

Hence

$$|\mathbf{u}(t) - \tilde{\mathbf{u}}(t)|_2 = |\psi(t)|_2 \exp\left(W(t) - \frac{t(b+\epsilon)}{2}\right) \to 0, \text{as } t \to \infty, \mathbb{P} - a.s.\text{on } B_R.$$

By the above asymptotic coupling theorem, the desired result follows.□

Remark about the method

By a bit more work, one can extend the result to the case of additive noise. The method of showing the existence and uniqueness of invariant measures can also be utilised to other highly non-linear hydrodynamical equations with or without pressure, for example 3D primitive equations and 2D nematic liquid flows, etc.

Thank you very much for listening!